L15: Tests with two-sided alternative hypotheses

1. Initial $\phi(X)$
   For $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ seeking desired properties $\alpha = \beta_\phi(\theta_0)$ and $[\beta_\phi(\theta_0)]'_\theta = 0$ for $\phi(X)$, with pdf/pmf $f(x, \theta)$ for sample X and $\theta_1 \neq \theta_0$ let

$$\int_x \phi(x)f(x;\,\theta_0)\,dx = \alpha,\ \int_x \phi(X)f'_\theta(x;\,\theta_0)\,dx = 0 \text{ and}$$
$$\phi(X) = \begin{cases} 1 & f(X;\,\theta_1) - k_1 f(X:\,\theta_0) - k_2 f'_\theta(X;\,\theta_0) > 0 \\ r & f(X;\,\theta_1) - k_1 f(X:\,\theta_0) - k_2 f'_\theta(X;\,\theta_0) = 0 \\ 0 & f(X;\,\theta_1) - k_1 f(X:\,\theta_0) - k_2 f'_\theta(X;\,\theta_0) < 0 \end{cases}$$

   Then by generalized Neyman-Pearson lemma

$$\int_x \psi(x)f(x;\,\theta_0)\,dx \le \alpha \text{ and } \int_x \psi(x)f'_\theta(x;\,\theta_0)\,dx \le 0 \text{ imply } E_{\theta_1}(\psi) \le E_{\theta_1}(\phi).$$

2. Modifications

   (1) An assumption and using sufficient statistic T

   Assume $f(x;\,\theta) = \exp[p(\theta) + q(x) + \theta T(x)]$ is the pdf/pmf for sample X.
   Then T is sufficient for θ and $f(t;\,\theta) = a(\theta)b(t)\exp(\theta t)$ where $a(\theta) > 0$ and $b(t) > 0$.
   Because all information on θ is contained in T, so X can be replaced by T. Thus

$$\phi(T) = \begin{cases} 1 & f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) > 0 \\ r & f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) = 0 \\ 0 & f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) < 0 \end{cases}$$

   where $\int_t \phi(t)f(t;\,\theta_0)\,dt = \alpha$ and $\int_t \phi(t)f'_\theta(t;\,\theta_0)\,dt = 0$.

   (2) Replacing $f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0)$

$$\begin{aligned} g_1(t;\,\theta_0,\,\theta_1) &= f(t;\,\theta_1) - k_1 f(t:\,\theta_0) - k_2 f'_\theta(t;\,\theta_0) \\ &= a(\theta_1)b(t)e^{\theta_1 t} - k_1 a(\theta_0)b(t)e^{\theta_0 t} - k_2[a(\theta_0)b(t)e^{\theta_0 t}t + a'(\theta_0)b(t)e^{\theta_0 t}] \end{aligned}$$

   So $g_1(t;\,\theta_0,\,\theta_1) > (=<)0 \Longleftrightarrow g_2(t;\,\theta_0,\,\theta_1) = \frac{a(\theta_1)}{a(\theta_0)}e^{(\theta_1-\theta_0)t} - k_1 - k_2\left(t + \frac{a'(\theta_0)}{a(\theta_0)}\right) > (=<)0$.

   But $[g_2]'_t = \frac{a(\theta_1)}{a(\theta_0)}(\theta_1 - \theta_0)e^{(\theta_1-\theta_0)t} - k_2$ and $[g_2]''_{t^2} = \frac{a(\theta_1)}{a(\theta_0)}(\theta_1 - \theta_0)^2 e^{(\theta_1-\theta_0)t} > 0$.

   Thus g_1 is a convex function of t. Hence there exist $c_1(\theta_1)$ and $c_2(\theta_1)$ such that

$$\begin{cases} f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) > 0 \\ f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) = 0 \\ f(T;\,\theta_1) - k_1 f(T:\,\theta_0) - k_2 f'_\theta(T;\,\theta_0) < 0 \end{cases} \Longleftrightarrow \begin{cases} T < c_1(\theta_1) & \text{or} & T > c_2(\theta_1) \\ T = c_1(\theta_1) & \text{or} & T = c_2(\theta_1) \\ c_1(\theta_1) < & T & < c_2(\theta_1) \end{cases}$$

   So

$$\phi(T) = \begin{cases} 1 & T < c_1(\theta_1) & \text{or} & T > c_2(\theta_1) \\ r & T = c_1(\theta_1) & \text{or} & T = c_2(\theta_1) \\ 0 & c_1(\theta_1) < & T & < c_2(\theta_1) \end{cases}$$

   where $\int_t \phi(t)f(t;\,\theta_0)\,dt = \alpha$ and $\int_t \phi(t)f'_\theta(t;\,\theta_0)\,dt = 0$.

(3) $\phi(T)$ is invariant over $\theta_1 \neq \theta_0$

$c_1(\theta_1$ and $c_2(\theta_2)$ in $\phi(T)$ are determined by

$$\int_t \phi(t) f(t;\,\theta_0)\,dt = \alpha \text{ and } \int_t \phi(t) f'_\theta(t;\,\theta_0)\,dt = 0.$$

But $f(t;\,\theta_0)$ and $f'_\theta(t;\,\theta_0)$ are invariant over all $\theta_1 \neq \theta_0$.
Hence $c_1(\theta_1)$ and $c_2(\theta_1)$ are free of θ_1. Thus

$$\phi(T) = \begin{cases} 1 & T < c_1 \quad \text{or} \quad T > c_2 \\ r & T = c_1 \quad \text{or} \quad T = c_2 \\ 0 & c_1 < \quad T \quad < c_2 \end{cases}$$

where $\int_t \phi(t) f(t;\,\theta_0)\,dt = \alpha$ and $\int_t \phi(t) f'_\theta(t;\,\theta_0)\,dt = 0$

3. UMPs

(1) An UMP in test class $\mathcal{C}$
For $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ suppose the pdf/pmf for sample X is $f(x;\,\theta) = \exp[p(\theta) + q(x) + \theta t(x)]$. Let $\mathcal{C}$ be $\mathcal{C} = \{\psi : E_{\theta_0}(\psi) \leq \alpha \text{ and } [E_{\theta_0}(\psi)]'_\theta = 0\}$. Then $\phi(T)$ in (3) of 2 is UMP test in $\mathcal{C}$.

Proof. Fist $\phi(T)$ is in $\mathcal{C}$.
Secondly if $\psi(T) \in \mathcal{C}$, by generalized Neyman-Pearson lemma

$$E_\theta(\psi) \leq E_\theta(\psi) \text{ for all } \theta \neq \theta_0.$$

(2) α-level unbiased UMP
For $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$ suppose the pdf/pmf for sample X is $f(x;\,\theta) = \exp[p(\theta) + q(x) + \theta t(x)]$. Then $\phi(T)$ in (3) of 2 is UMP test in the class of all α-level unbiased tests with differentiable mean of critical functions.

Proof. First $\phi(T)$ is an α-level test.
Let $\psi \equiv \alpha$. Then $\psi \in \mathcal{C}$ in (1). But ϕ is UMP in $\mathcal{C}$. Thus

$$E_{\theta_0}(\phi) = \alpha = E_\theta(\psi) \leq E_\theta(\phi) \text{ for all } \theta \neq \theta_0.$$

So ϕ is also an UB test. Hence ϕ is an α-level unbiased test.
If ψ is also α-level UB test with differentiable mean, then

$$\int_t \psi(t) f(t;\,\theta_0)\,dt \leq \alpha \text{ and } \int_t \psi(t) f'_\theta(t;\,\theta_0)\,dt = 0.$$

Therefore $\psi \in \mathcal{C}$, So

$$E_\theta(\psi) \leq E_\theta(\phi) \text{ for all } \theta \neq \theta_0.$$